Mechanics of Composite Materials

Version 2.1

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Assumed Pre-knowledge

It is assumed that the student is familiar with simple concepts of mechanical behaviour, such as the broad meanings of stress and strain. It would be an advantage for the student to understand that these are really tensor quantities, although this is by no means essential. All of the terms associated with the assumed pre-knowledge are defined in the glossary, which can be consulted by the student at any time.

Most of the material in this package is based on a recently published book. This is:


This source should be consulted for background to the treatments in this module, particularly mathematical details.

What is a Composite Material?

Most composites have strong, stiff fibres in a matrix which is weaker and less stiff. The objective is usually to make a component which is strong and stiff, often with a low density. Commercial material commonly has glass or carbon fibres in matrices based on thermosetting polymers, such as epoxy or polyester resins. Sometimes, thermoplastic polymers may be preferred, since they are mouldable after initial production. There are further classes of composite in which the matrix is a metal or a ceramic. For the most part, these are still in a developmental stage, with problems of high manufacturing costs yet to be overcome. Furthermore, in these composites the reasons for adding the fibres (or, in some cases, particles) are often rather complex; for example, improvements may be sought in creep, wear, fracture toughness, thermal stability, etc. This software package covers simple mechanics concepts of stiffness and strength, which, while applicable to all composites, are often more relevant to fibre-reinforced polymers.

Module Structure

The module comprises three sections:

- Load Transfer
- Composite Laminates
- Fracture Behaviour

Brief descriptions are given below of the contents of these sections, covering both the main concepts involved and the structure of the software.

Load Transfer
Summary

This section covers basic ideas concerning the manner in which an applied mechanical load is shared between the matrix and the fibres. The treatment starts with the simple case of a composite containing aligned, continuous fibres. This can be represented by the slab model. For loading parallel to the fibre axis, the equal strain condition is imposed, leading to the Rule of Mixtures expression for the Young's modulus. This is followed by the cases of transverse loading of a continuous fibre composite and axial loading with discontinuous fibres.

What is meant by Load Transfer?

The concept of load sharing between the matrix and the reinforcing constituent (fibre) is central to an understanding of the mechanical behaviour of a composite. An external load (force) applied to a composite is partly borne by the matrix and partly by the reinforcement. The load carried by the matrix across a section of the composite is given by the product of the average stress in the matrix and its sectional area. The load carried by the reinforcement is determined similarly. Equating the externally imposed load to the sum of these two contributions, and dividing through by the total sectional area, gives a basic and important equation of composite theory, sometimes termed the "Rule of Averages".

\[ f\bar{\sigma}_m + (1-f)\bar{\sigma}_f = \sigma_A \]  

which relates the volume-averaged matrix and fibre stresses (\(\bar{\sigma}_m\), \(\bar{\sigma}_f\)), in a composite containing a volume (or sectional area) fraction \(f\) of reinforcement, to the applied stress \(\sigma_A\).

Thus, a certain proportion of an imposed load will be carried by the fibre and the remainder by the matrix. Provided the response of the composite remains elastic, this proportion will be independent of the applied load and it represents an important characteristic of the material. It depends on the volume fraction, shape and orientation of the reinforcement and on the elastic properties of both constituents. The reinforcement may be regarded as acting efficiently if it carries a relatively high proportion of the externally applied load. This can result in higher strength, as well as greater stiffness, because the reinforcement is usually stronger, as well as stiffer, than the matrix.

What happens when a Composite is Stressed?

Consider loading a composite parallel to the fibres. Since they are bonded together, both fibre and matrix will stretch by the same amount in this direction, i.e. they will have equal strains, \(\varepsilon\) (Fig. 1). This means that, since the fibres are stiffer (have a higher Young modulus, \(E\)), they will be carrying a larger stress. This illustrates the concept of load transfer, or load...
**partitioning** between matrix and fibre, which is desirable since the fibres are better suited to bear high stresses. By putting the sum of the contributions from each phase equal to the overall load, the Young modulus of the composite is found (diagram). It can be seen that a "Rule of Mixtures" applies. This is sometimes termed the "equal strain" or "Voigt" case. Page 2 in the section covers derivation of the equation for the axial stiffness of a composite and page 3 allows the effects on composite stiffness of the fibre/matrix stiffness ratio and the fibre volume fraction to be explored by inputting selected values.

**What about the Transverse Stiffness?**

Also of importance is the response of the composite to a load applied transverse to the fibre direction. The stiffness and strength of the composite are expected to be much lower in this case, since the (weak) matrix is not shielded from carrying stress to the same degree as for axial loading. Prediction of the transverse stiffness of a composite from the elastic properties of the constituents is far more difficult than the axial value. The conventional approach is to assume that the system can again be represented by the "slab model". A lower bound on the stiffness is obtained from the "equal stress" (or "Reuss") assumption shown in Fig. 2. The value is an underestimate, since in practice there are parts of the matrix effectively "in parallel" with the fibres (as in the equal strain model), rather than "in series" as is assumed. Empirical expressions are available which give much better approximations, such as that of Halpin-Tsai. There are again two pages in the section covering this topic, the first (page 4) outlining derivation of the equal stress equation for stiffness and the second (page 5) allowing this to be evaluated for different cases. For purposes of comparison, a graph is plotted of equal strain, equal stress and Halpin-Tsai predictions. The Halpin-Tsai expression for transverse stiffness (which is not given in the module, although it is available in the glossary) is:

\[
E_2 = \frac{E_m(1 + \xi \eta f)}{(1 - \eta f)}
\]  

(2)

in which

\[
\eta = \left( \frac{E_f}{E_m} - 1 \right) \left( \frac{E_f}{E_m} + \xi \right)
\]
The value of $x$ may be taken as an adjustable parameter, but its magnitude is generally of the order of unity. The expression gives the correct values in the limits of $f=0$ and $f=1$ and in general gives good agreement with experiment over the complete range of fibre content. A general conclusion is that the transverse stiffness (and strength) of an aligned composite are poor; this problem is usually countered by making a laminate (see section on "composite laminates").

**How is Strength Determined?**

There are several possible approaches to prediction of the strength of a composite. If the stresses in the two constituents are known, as for the long fibre case under axial loading, then these values can be compared with the corresponding strengths to determine whether either will fail. Page 6 in the section briefly covers this concept. (More details about strength are given in the section on "Fracture Behaviour"). The treatment is a logical development from the analysis of axial stiffness, with the additional input variable of the ratio between the strengths of fibre and matrix.

Such predictions are in practice complicated by uncertainties about in situ strengths, interfacial properties, residual stresses etc. Instead of relying on predictions such as those outlined above, it is often necessary to measure the strength of the composite, usually by loading parallel, transverse and in shear with respect to the fibres. This provides a basis for prediction of whether a component will fail when a given set of stresses is generated (see section on "Fracture Behaviour"), although in reality other factors such as environmental degradation or the effect of failure mode on toughness, may require attention.

**What happens with Short Fibres?**

Short fibres can offer advantages of economy and ease of processing. When the fibres are not long, the equal strain condition no longer holds under axial loading, since the stress in the fibres tends to fall off towards their ends (see Fig. 3). This means that the average stress in the matrix must be higher than for the long fibre case. The effect is illustrated pictorially in pages 7 and 8 of the section.

![Figure 3](image)

This lower stress in the fibre, and correspondingly higher average stress in the matrix (compared with the long fibre case) will depress both the stiffness and strength of the composite, since the matrix is both weaker and less stiff than the fibres. There is therefore
interest in quantifying the change in stress distribution as the fibres are shortened. Several models are in common use, ranging from fairly simple analytical methods to complex numerical packages. The simplest is the so-called "shear lag" model. This is based on the assumption that all of the load transfer from matrix to fibre occurs via shear stresses acting on the cylindrical interface between the two constituents. The build-up of tensile stress in the fibre is related to these shear stresses by applying a force balance to an incremental section of the fibre. This is depicted in page 9 of the section. It leads to an expression relating the rate of change of the stress in the fibre to the interfacial shear stress at that point and the fibre radius, \( r \).

\[
\frac{d\sigma_f}{dx} = -\frac{2\tau_i}{r}
\]

which may be regarded as the basic shear lag relationship. The stress distribution in the fibre is determined by relating shear strains in the matrix around the fibre to the macroscopic strain of the composite. Some mathematical manipulation leads to a solution for the distribution of stress at a distance \( x \) from the mid-point of the fibre which involves hyperbolic trig functions:

\[
\sigma_f = E_f e_1 \left[ 1 - \cosh(n\pi r) \text{sech}(ns) \right]
\]

where \( e_1 \) is the composite strain, \( s \) is the fibre aspect ratio (length/diameter) and \( n \) is a dimensionless constant given by:

\[
n = \left( \frac{2 E_m}{E_f (1 + n_m) \ln(1 / f)} \right)^{1/2}
\]

in which \( n_m \) is the Poisson ratio of the matrix. The variation of interfacial shear stress along the fibre length is derived, according to Eq.(3), by differentiating this equation, to give:

\[
\tau_i = \frac{n e_1}{2} E_f \sinh\left( \frac{n\pi r}{r} \right) \text{sech}(ns)
\]

The equation for the stress in the fibre, together with the assumption of a average tensile strain in the matrix equal to that imposed on the composite, can be used to evaluate the composite stiffness. This leads to:

\[
\sigma_1 = e_1 \left[ \frac{E_f}{E_m} \left( 1 - \frac{\tanh(ns)}{ns} \right) + (1 - f) E_m \right]
\]

The expression in square brackets is the composite stiffness. In page 10 of the section, there is an opportunity to examine the predicted stiffness as a function of fibre aspect ratio, fibre/matrix stiffness ratio and fibre volume fraction. The other point to note about the shear lag model is that it can be used to examine inelastic behaviour. For example, interfacial sliding (when the interfacial shear stress reaches a critical value) or fibre fracture (when the tensile stress in the fibre becomes high enough) can be predicted. As the strain imposed on the
composite is increased, sliding spreads along the length of the fibre, with the interfacial shear stress unable to rise above some critical value, $t^i$. If the interfacial shear stress becomes uniform at $t^*i$ along the length of the fibre, then a critical aspect ratio, $s^*$, can be identified, below which the fibre cannot undergo fracture. This corresponds to the peak (central) fibre stress just attaining its ultimate strength $sf^*$, so that, by integrating Eq.(3) along the fibre half-length:

It follows from this that a distribution of aspect ratios between $s^*$ and $s^*/2$ is expected, if the composite is subjected to a large strain. The value of $s^*$ ranges from over 100, for a polymer composite with poor interfacial bonding, to about 2-3 for a strong metallic matrix. In page 10, the effects of changing various parameters on the distributions of interfacial shear stress and fibre tensile stress can be explored and predictions made about whether fibres of the specified aspect ratio can be loaded up enough to cause them to fracture.

**Conclusion**

After completing this section, the student should:

- Appreciate that the key issue, controlling both stiffness and strength, is the way in which an applied load is shared between fibres and matrix.
- Understand how the slab model is used to obtain axial and transverse stiffnesses for long fibre composites.
- Realise why the slab model (equal stress) expression for transverse stiffness is an underestimate and be able to obtain a more accurate estimate by using the Halpin-Tsai equation.
- Understand broadly why the axial stiffness is lower when the fibres are discontinuous and appreciate the general nature of the stress field under load in this case.
- Be able to use the shear lag model to predict axial stiffness and to establish whether fibres of a given aspect ratio can be fractured by an applied load.
- Note that the treatments employed neglect thermal residual stresses, which can in practice be significant in some cases.

**Summary**

This section covers the advantages of lamination, the factors affecting choice of laminate structure and the approach to prediction of laminate properties. It is first confirmed that, while unidirectional plies can have high axial stiffness and strength, these properties are markedly anisotropic. With a laminate, there is scope for tailoring the properties in different directions within a plane to the requirements of the component. Both elastic and strength properties can be predicted once the stresses on the individual plies have been established. This is done by first studying how the stiffness of a ply depends on the angle between the loading direction and then imposing the condition that all the individual plies in a laminate must exhibit the same strain. The methodology for prediction of the properties of any laminate is thus outlined, although most of the mathematical details are kept in the background.

**What is a Laminate?**
High stiffness and strength usually require a high proportion of fibres in the composite. This is achieved by aligning a set of long fibres in a thin sheet (a lamina or ply). However, such material is highly anisotropic, generally being weak and compliant (having a low stiffness) in the transverse direction. Commonly, high strength and stiffness are required in various directions within a plane. The solution is to stack and weld together a number of sheets, each having the fibres oriented in different directions. Such a stack is termed a laminate. An example is shown in the diagram. The concept of a laminate, and a pictorial illustration of the way that the stiffness becomes more isotropic as a single ply is made into a cross-ply laminate, are presented in page 1 of this section.

**What are the Stresses within a Crossply Laminate?**

The stiffness of a single ply, in either axial or transverse directions, can easily be calculated. (See the section on Load Transfer). From these values, the stresses in a crossply laminate, when loaded parallel to the fibre direction in one of the plies, can readily be calculated. For example, the slab model can be applied to the two plies in exactly the same way as it was applied in the last section to fibres and matrix. This allows the stiffness of the laminate to be calculated. This gives the strain (experienced by both plies) in the loading direction, and hence the average stress in each ply, for a given applied stress. The stresses in fibre and matrix within each ply can also be found from these average stresses and a knowledge of how the load is shared. In page 2 of this section, by inputting values for the fibre/matrix stiffness ratio and fibre content, the stresses in both plies, and in their constituents, can be found. Note that, particularly with high stiffness ratios, most of the applied load is borne by the fibres in the "parallel" ply (the one with the fibre axis parallel to the loading axis).

**What is the Off-Axis Stiffness of a Ply?**

For a general laminate, however, or a crossply loaded in some arbitrary direction, a more systematic approach is needed in order to predict the stiffness and the stress distribution. Firstly, it is necessary to establish the stiffness of a ply oriented so the fibres lie at some arbitrary angle to the stress axis. Secondly, further calculation is needed to find the stiffness of a given stack. Consider first a single ply. The stiffness for any loading angle is evaluated as follows, considering only stresses in the plane of the ply The applied stress is first transformed to give the components parallel and perpendicular to the fibres. The strains generated in these directions can be calculated from the (known) stiffness of the ply when referred to these axes. Finally, these strains are transformed to values relative to the loading direction, giving the stiffness.
These three operations can be expressed mathematically in tensor equations. Since we are only concerned with stresses and strains within the plane of the ply, only 3 of each (two normal and one shear) are involved. The first step of resolving the applied stresses, $s_x$, $s_y$ and $t_{xy}$, into components parallel and normal to the fibre axis, $s_1$, $s_2$ and $t_{12}$ (see Fig. 4), depends on the angle, $f$ between the loading direction ($x$) and the fibre axis (1)

$$\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\gamma_{12}
\end{bmatrix} = |T| \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\gamma_{xy}
\end{bmatrix}$$

(9)

where the transformation matrix is given by:

$$|T| = \begin{vmatrix}
c^2 & s^2 & 2cs \\
s^2 & c^2 & -2cs \\
cos \phi & sin \phi & c^2 - s^2
\end{vmatrix}$$

(10)

in which $c = \cos f$ and $s = \sin f$. For example, the value of $s_1$ would be obtained from:

$$s_1 = \sigma_x \cos^2 \phi + \sigma_y \sin^2 \phi + 2 \tau_{xy} \cos \phi \sin \phi$$

(11)

Now, the elastic response of the ply to stresses parallel and normal to the fibre axis is easy to analyse. For example, the axial and transverse Young’s moduli ($E_1$ and $E_2$) could be obtained using the slab model or Halpin-Tsai expressions (see Load Transfer section). Other elastic constants, such as the shear modulus ($G_{12}$) and Poisson’s ratios, are readily calculated in a similar way. The relationship between stresses and resultant strains dictated by these elastic constants is neatly expressed by an equation involving the compliance tensor, $S$, which for our composite ply, has the form:

$$\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix} = |S| \begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\gamma_{12}
\end{bmatrix} = \begin{vmatrix}
S_{11} & S_{12} & 0 \\
S_{12} & S_{22} & 0 \\
0 & 0 & S_{66}
\end{vmatrix} \begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\gamma_{12}
\end{bmatrix}$$

(12)
in which, by inspection of the individual equations, it can be seen that

\[ S_{11} = \frac{1}{E_1} \]
\[ S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2} \]
\[ S_{22} = \frac{1}{E_2} \]
\[ S_{66} = \frac{1}{G_{12}} \]

Application of Eq.(12), using the stresses established from Eq.(9), now allows the strains to be established, relative to the 1 and 2 directions. There is a minor complication in applying the final stage of converting these strains so that they refer to the direction of loading (x and y axes). Because engineering and tensorial shear strains are not quite the same, a slightly different transformation matrix is applicable from that used for stresses

\[ \varepsilon_x = \frac{\sigma_x}{2\tau_{1x}} \]
\[ \begin{vmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{vmatrix} = |T^t| \begin{vmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{vmatrix} \]

in which,

\[ |T^t| = \begin{vmatrix} c^2 & s^2 & c\sigma \\ s^2 & c^2 & -c\sigma \\ c\gamma & c\gamma & c^2-\sigma^2 \end{vmatrix} \]

and the inverse of this matrix is used for conversion in the reverse direction,

\[ \begin{vmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{vmatrix} = |T^t|^{-1} \begin{vmatrix} \sigma_1 \\ \sigma_2 \\ \gamma_{12} \end{vmatrix} \]

in which,

\[ |T^t|^{-1} = \begin{vmatrix} c^2 & s^2 & -c\sigma \\ s^2 & c^2 & c\sigma \\ c\gamma & c\gamma & c^2+\sigma^2 \end{vmatrix} \]

The final expression relating applied stresses and resultant strains can therefore be written,
The elements of \(| \mathbf{S} \)|, the transformed compliance tensor, are obtained by concatenation (the equivalent of multiplication) of the matrices \(| \mathbf{T} \mathbf{T}^\dagger |, \mathbf{| S |,} \mathbf{| T |.} \) The following expressions are obtained

\[
\begin{align*}
\bar{S}_{11} &= S_{11} c^4 + S_{22} s^4 + 2S_{12} + S_{66} c^2 s^2 \\
\bar{S}_{12} &= S_{12} \left( c^4 + s^4 + S_{11} + S_{22} - S_{66} \right) c^2 s^2 \\
\bar{S}_{22} &= S_{11} s^4 + S_{22} c^4 + 2S_{12} + S_{66} c^2 s^2 \\
\bar{S}_{16} &= (2S_{11} - 2S_{12} - S_{66}) c^3 s - (2S_{22} - 2S_{12} - S_{66}) c s^3 \\
\bar{S}_{26} &= (2S_{11} - 2S_{12} - S_{66}) c s^3 - (2S_{22} - 2S_{12} - S_{66}) c^3 s \\
\bar{S}_{66} &= (4S_{11} + 4S_{22} - 8S_{12} - 2S_{66}) c^2 s^2 + S_{66} c^4 + s^4
\end{align*}
\]

The final result of this rather tedious derivation is therefore quite straightforward. Eq.(16), together with the elastic constants of the composite when loaded parallel and normal to the fibre axis, allows the elastic deformation of the ply to be predicted for loading at any angle to the fibre axis. This is conveniently done using a simple computer program. The results of such calculations can be explored using pages 4 and 5 in this section. As an example, Fig. 5 shows the Young's modulus for the an polyester-50% glass fibre ply as the angle, \(\theta\) between fibre axis and loading direction rises from \(0^\circ\) to \(90^\circ\). A sharp fall is seen as \(\theta\) exceeds about \(5-10^\circ\).

How is the Stiffness of a Laminate obtained?

Once the elastic response of a single ply loaded at an arbitrary angle has been established, that of a stack bonded together (i.e. a laminate) is quite easy to predict. For example, the Young’s modulus in the loading direction is given by an applied normal stress over the resultant normal strain in that direction. This same strain will be experienced by all of the component plies of the laminate. Since every ply now has a known Young’s modulus in the loading direction (dependent on its fibre direction), the stress in each one can be expressed in terms of this universal strain. Furthermore, the force (stress times sectional area) represented by the applied stress can also be expressed as the sum of the forces being carried by each ply. This allows the overall Young’s modulus of the laminate to be calculated. The results of such calculations, for any selected stacking sequence, can be explored using pages 4 and 5.

Are Other Elastic Constants Important?

There are several points of interest about how a ply changes shape in response to an applied load. For example, the lateral contraction (Poisson ratio, \(\nu\)) behaviour may be important, since in a laminate such contraction may be resisted by other plies, setting up stresses
transverse to the applied load. Another point with fibre composites under off-axis loading is that shear strains can arise from tensile stresses (and vice versa). This corresponds to the elements of $S$ which are zero in Eq.(12) becoming non-zero for an arbitrary loading angle (Eq.(16)). These so-called "tensile-shear interactions" can be troublesome, since they can set up stresses between individual plies and can cause the laminate to become distorted. The value of $\tilde{g}_{12}$, for example, represents the ratio between $g_{12}$ and $s_1$. Its value can be obtained for any specified laminate by using page 6 of this section. It will be seen that, depending on the stacking sequence, relatively high distortions of this type can arise. On the other hand, a stacking sequence with a high degree of rotational symmetry can show no tensile-shear interactions. When the tensile-shear interaction terms contributed by the individual laminae all cancel each other out in this way, the laminate is said to be "balanced". Simple crossply and angle-ply laminates are not balanced for a general loading angle, although both will be balanced when loaded at $f=0^\circ$ (i.e., parallel to one of the plies for a cross-ply or equally inclined to the $+q$ and $-q$ plies for the angle-ply case). If the plies vary in thickness, or in the volume fractions or type of fibres they contain, then even a laminate in which the stacking sequence does exhibit the necessary rotational symmetry is prone to tensile-shear distortions and computation is necessary to determine the lay-up sequence required to construct a balanced laminate. The stacking order in which the plies are assembled does not enter into these calculations.

**Conclusion**

After completing this section, the student should:

- Appreciate that, while individual plies are highly anisotropic, they can be assembled into laminates having a selected set of in-plane properties.
- Understand broadly how the elastic properties of a laminate, and the partitioning of an applied load between the constituent plies, can be predicted.
- Be able to use the software package to predict the characteristics of specified laminate structures.
- Understand the meaning of a "balanced" laminate.

**Fracture Behaviour**

**Summary**

This section covers simple approaches to prediction of the failure of composites from properties of matrix and fibre and from interfacial characteristics. The axial strength of a continuous fibre composite can be predicted from properties of fibre and matrix when tested in isolation. Failures when loaded transversely or in shear relative to the fibre direction, on the other hand, tends to be sensitive to the interfacial strength and must therefore be measured experimentally. An outline is given of how these measured strengths can be used to predict failure of various laminate structures made from the composite concerned. Finally, a brief description is given of what is meant by the toughness (fracture energy) of a material. In composites the most significant contribution to the fracture energy usually comes from fibre pullout. A simple model is presented for prediction of the fracture energy from fibre pullout, depending on fibre aspect ratio, fibre radius and interfacial shear strength.

**How do Composites Fracture?**
Fracture of long fibre composites tends to occur either normal or parallel to the fibre axis. This is illustrated on page 1 of this section - see Fig. 6. Large tensile stresses parallel to the fibres, $s_1$, lead to fibre and matrix fracture, with the fracture path normal to the fibre direction. The strength is much lower in the transverse tension and shear modes and the composite fractures on surfaces parallel to the fibre direction when appropriate $s_2$ or $t_{12}$ stresses are applied. In these cases, fracture may occur entirely within the matrix, at the fibre/matrix interface or primarily within the fibre. To predict the strength of a lamina or laminate, values of the failure stresses $s_1^*$, $s_2^*$ and $t_{12}^*$ have to be determined.

**Can the Axial Strength be Predicted?**

Understanding of failure under an applied tensile stress parallel to the fibres is relatively simple, provided that both constituents behave elastically and fail in a brittle manner. They then experience the same axial strain and hence sustain stresses in the same ratio as their Young’s moduli. Two cases can be identified, depending on whether matrix or fibre has the lower strain to failure. These cases are treated in pages 2 and 3 respectively.
Consider first the situation when the matrix fails first ($e_m^* < e_f^*$). For strains up to $e_m^*$, the composite stress is given by the simple rule of mixtures:

$$\sigma_1 = f \sigma_f + (1-f) \sigma_m$$ (17)

Above this strain, however, the matrix starts to undergo microcracking and this corresponds with the appearance of a "knee" in the stress-strain curve. The composite subsequently extends with little further increase in the applied stress. As matrix cracking continues, the load is transferred progressively to the fibres. If the strain does not reach $e_f^*$ during this stage, further extension causes the composite stress to rise and the load is now carried entirely by the fibres. Final fracture occurs when the strain reaches $e_f^*$, so that the composite failure stress $s_1^*$ is given by $f s_f^*$. A case like this is illustrated in Fig. 7, which refers to steel rods in a concrete matrix.

[F162, RHS, real system data, mild steel fibres, concrete matrix, fibre fraction 40%, "strength v. fraction of fibres" clicked]
Alternatively, if the fibres break before matrix cracking has become sufficiently extensive to transfer all the load to them, then the strength of the composite is given by:

$$\sigma_{1*} = f \sigma_{fm*} + (1-f) \sigma_{m*}$$

(18)

where $\sigma_{fm*}$ is the fibre stress at the onset of matrix cracking ($e_1=e_{m*}$). The composite failure stress depends therefore on the fibre volume fraction in the manner shown in Fig. 8. The fibre volume fraction above which the fibres can sustain a fully transferred load is obtained by setting the expression in Eq.(18) equal to $f \sigma_{m*}$, leading to:

$$f' = \frac{\sigma_{m*}}{\sigma_{fm*} - \sigma_{fm*} + \sigma_{m*}}$$

(19)

If the fibres have the smaller failure strain (page 3), continued straining causes the fibres to break up into progressively shorter lengths and the load to be transferred to the matrix. This continues until all the fibres have aspect ratios below the critical value (see Eq.(8)). It is often assumed in simple treatments that only the matrix is bearing any load by the time that break-up of fibres is complete. Subsequent failure then occurs at an applied stress of $(1-f) \sigma_{m*}$. If matrix fracture takes place while the fibres are still bearing some load, then the composite failure stress is:

$$\sigma_{1*} = f \sigma_{f*} + (1-f) \sigma_{mf*}$$

(20)

where $\sigma_{mf}$ is the matrix stress at the onset of fibre cracking. In principle, this implies that the presence of a small volume fraction of fibres reduces the composite failure stress below that of the unreinforced matrix. This occurs up to a limiting value $f'$ given by setting the right hand side of Eq.(20) equal to $(1-f) \sigma_{m*}$. 

Figure 8
The values of these parameters can be explored for various systems using pages 2 and 3. Prediction of the values of $s_{2^*}$ and $t_{12^*}$ from properties of the fibre and matrix is virtually impossible, since they are so sensitive to the nature of the fibre-matrix interface. In practice, these strengths have to be measured directly on the composite material concerned.

**How do Plies Fail under Off-axis Loads?**

Failure of plies subjected to arbitrary (in-plane) stress states can be understood in terms of the three failure mechanisms (with defined values of $s_{1^*}$, $s_{2^*}$ and $t_{12^*}$) which were depicted on page 1. A number of failure criteria have been proposed. The main issue is whether or not the critical stress to trigger one mechanism is affected by the stresses tending to cause the others - i.e. whether there is any interaction between the modes of failure. In the simple maximum stress criterion, it is assumed that failure occurs when a stress parallel or normal to the fibre axis reaches the appropriate critical value, that is when one of the following is satisfied:

\[
\sigma_1 \geq \sigma_{1^*} \quad \sigma_2 \geq \sigma_{2^*} \quad \tau_{12} \geq \tau_{12^*}
\]  

For any stress system ($\sigma_x$, $\sigma_y$ and $\tau_{xy}$) applied to the ply, evaluation of these stresses can be carried out as described in the section on Composite Laminates (Eqs.(9) and (10)).
Monitoring of $s_1$, $s_2$ and $t_{12}$ as the applied stress is increased allows the onset of failure to be identified as the point when one of the inequalities in Eq.(22) is satisfied. Noting the form of $|\mathbf{T}|$ (Eq.(10)), and considering applied uniaxial tension, the magnitude of $s_1$ necessary to cause failure can be plotted as a function of angle $\phi$ between stress axis and fibre axis, for each of the three failure modes.

\[
\sigma_{1*} = \frac{\sigma_{1*}}{\cos^2 \phi},
\]

\[
\sigma_{2*} = \frac{\sigma_{2*}}{\sin^2 \phi},
\]

\[
\sigma_{t*} = \frac{\tau_{12*}}{\sin \phi \cos \phi}.
\]

The applied stress levels at which these conditions become satisfied can be explored using page 5. As an example, the three curves corresponding to Eqs.(23)-(25) are plotted in Fig. 9, using typical values of $s_{1*}$, $s_{2*}$ and $t_{12*}$. Typically, axial failure is expected only for very small loading angles, but the predicted transition from shear to transverse failure may occur anywhere between 20° and 50°, depending on the exact values of $t_{12*}$ and $s_{2*}$.

In practice, there is likely to be some interaction between the failure modes. For example, shear failure is expected to occur more easily if, in addition to the shear stress, there is also a normal
tensile stress acting on the shear plane. The most commonly used model taking account of this effect is the Tsai-Hill criterion. This can be expressed mathematically as

\[
\left( \frac{\sigma_1}{\sigma_{\text{P}}} \right)^2 + \left( \frac{\sigma_2}{\sigma_{\text{P}}} \right)^2 - \frac{\sigma_1 \sigma_2}{\sigma_{\text{P}}^2} + \left( \frac{\tau_{12}}{\tau_{12}^*} \right)^2 - 1
\]

(26)

This defines an envelope in stress space: if the stress state \((\sigma_1, \sigma_2, \tau_{12})\) lies outside of this envelope, i.e. if the sum of the terms on the left hand side is equal to or greater than unity, then failure is predicted. The failure mechanism is not specifically identified, although inspection of the relative magnitudes of the terms in Eq.(26) gives an indication of the likely contribution of the three modes. Under uniaxial loading, the Tsai-Hill criterion tends to give rather similar predictions to the Maximum Stress criterion for the strength as a function of loading angle. The predicted values tend to be somewhat lower with the Tsai-Hill criterion, particularly in the mixed mode regimes where both normal and shear stresses are significant. This can be explored on page 6.

**What is the Failure Strength of a Laminate?**

The strength of laminates can be predicted by an extension of the above treatment, taking account of the stress distributions in laminates, which were covered in the preceding section. Once these stresses are known (in terms of the applied load), an appropriate failure criterion can be applied and the onset and nature of the failure predicted.

<table>
<thead>
<tr>
<th>PLY</th>
<th>(\theta)</th>
<th>Fibre</th>
<th>Matrix</th>
<th>f(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
<td>glass E</td>
<td>epoxy</td>
<td>50</td>
</tr>
<tr>
<td>2</td>
<td>-30</td>
<td>glass E</td>
<td>epoxy</td>
<td>50</td>
</tr>
</tbody>
</table>

![Figure 10](image)

However, failure of an individual ply within a laminate does not necessarily mean that the
component is no longer usable, as other plies may be capable of withstanding considerably greater loads without catastrophic failure. Analysis of the behaviour beyond the initial, fully elastic stage is complicated by uncertainties as to the degree to which the damaged plies continue to bear some load. Nevertheless, useful calculations can be made in this regime (although the major interest may be in the avoidance of any damage to the component). In page 7, a crossply (0/90) laminate is loaded in tension along one of the fibre directions. The stresses acting in each ply, relative to the fibre directions, are monitored as the applied stress is increased. Only transverse or axial tensile failure is possible in either ply, since no shear stresses act on the planes parallel to the fibre directions. The software allows the onset of failure to be predicted for any given composite with specified strength values. Although the parallel ply takes most of the load, it is commonly the transverse ply which fails first, since its strength is usually very low.

<table>
<thead>
<tr>
<th>PLY</th>
<th>( \sigma_1 )</th>
<th>( \sigma_2 )</th>
<th>( \tau_{12} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>252.42</td>
<td>-27.42</td>
<td>-49.12</td>
</tr>
<tr>
<td>2</td>
<td>252.42</td>
<td>-27.42</td>
<td>49.12</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

In page 8, any specified laminate can be subjected to an imposed stress state and the onset of failure predicted. An example of such a calculation is shown in Fig. 10.

What is the Toughness (Fracture Energy) of a Composite?

The fracture energy, \( G_c \), of a material is the energy absorbed within it when a crack advances through the section of a specimen by unit area. Potentially the most significant source of fracture work for most fibre composites is interfacial frictional sliding. Depending on the interfacial roughness, contact pressure and sliding distance, this process can absorb large quantities of energy. The case of most interest is pull-out of fibres from their sockets in the matrix. This process is illustrated schematically in page 9.

The work done as a crack opens up and fibres are pulled out of their sockets can be calculated in the following way. A simple shear lag approach is used. Provided the fibre aspect ratio, \( s (=L/r) \), is less than the critical value, \( s^* (=s_{fr}/2t_i) \), see page 10 of the Load Transfer section, all of the fibres intersected by the crack debond and are subsequently pulled out of their sockets in the matrix (rather than fracturing). Consider a fibre with a remaining embedded length of \( x \) being pulled out an increment of distance \( dx \). The associated work is given by the product of the force acting on the fibre and the distance it moves

\[
dU = (2\pi r x t_i^*) \, dx
\]

(27)

where \( t_i^* \) is the interfacial shear stress, taken here as constant along the length of the fibre. The work done in pulling this fibre out completely is therefore given by
where $x_0$ is the embedded length of the fibre concerned on the side of the crack where debonding occurs ($x_0 = L$). The next step is an integration over all of the fibres. If there are $N$ fibres per m$^2$, then there will be ($N dx_0 / L$) per m$^2$ with an embedded length between $x_0$ and $(x_0 + dx_0)$. This allows an expression to be derived for the pull-out work of fracture, $G_c$

$$
\Delta U = \int_0^{x_0} 2p r x \tau_i, \ dx = pr x^2 \tau_i \ (28)
$$

This contribution to the overall fracture energy can be large. For example, taking $f=0.5$, $s=50$, $r=10 \ \mu m$ and $t_i*=20$ MPa gives a value of about 80 kJ m$^{-2}$. This is greater than the fracture energy of many metals. Since $s_f*$ would typically be about 3 GPa, the critical aspect ratio, $s^* = (s_f* / 2t_i*)$, for this value of $t_i*$, would be about 75. Since this is greater than the actual aspect ratio, pull-out is expected to occur (rather than fibre fracture), so the calculation should be valid. The pull-out energy is greater when the fibres have a larger diameter, assuming that the fibre aspect ratio is the same. In page 10, the cumulative fracture energy is plotted as the crack opens up and fibres are pulled out of their sockets. The end result for a particular case is shown in Fig. 11.
Conclusion

After completing this section, the student should:

- Appreciate that a unidirectional composite tends to fracture axially, transversely or in shear relative to the fibre direction.
- Be able to use simple expressions for axial composite strength, based on fibre and matrix fracturing similarly in the composite and in isolation.
- Understand what is meant by "mixed mode" failure and be able to use Maximum Stress or Tsai-Hill criteria to predict how a unidirectional composite will fail under multi-axial loading.
- Be able to use measured strength values for a unidirectional composite to predict how ply damage will develop in a laminate.
- Understand the concept of the fracture energy of a composite and be able to use the software package to predict the contribution to this from fibre pull-out.

Bibliography

The student is referred to the following resources in this module:

Chavla, K.K., *Ceramic Matrix Composites*, Chapman and Hall, 1993


